A powerful approach for fractional Drinfeld–Sokolov–Wilson equation with Mittag-Leffler law

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1. Introduction

Fractional calculus (FC) was originated in Newton’s time, but lately, it fascinated the attention of many scholars. From the last thirty years, the most intriguing leaps in scientific and engineering applications have been found within the framework of FC. The concept of the fractional derivative has been industrialized due to the complexities associated with a heterogeneities phenomenon. The fractional differential operator are capable to capture the behaviour of multifaceted media having diffusion process. It has been a very essential tool and many problems can be illustrated more conveniently and more accurately with differential equations having an arbitrary order. Due to the swift development of mathematical techniques with computer software’s, many researchers started to work on generalised calculus to present their viewpoints while analysing many complex phenomena.

Abstract The pivotal aim of the present work is to find the solution for fractional Drinfeld–Sokolov–Wilson equation using q-homotopy analysis transform method (q-HATM). The proposed technique is graceful amalgamations of Laplace transform technique with q-homotopy analysis scheme, and fractional derivative defined with Atangana-Baleanu (AB) operator. The fixed point hypothesis considered in order to demonstrate the existence and uniqueness of the obtained solution for the proposed fractional order model. In order to validate and illustrate the efficiency of the future technique, we analysed the projected model in terms of fractional order. Meanwhile, the physical behaviour of the q-HATM solutions have been captured in terms of plots for diverse fractional order and the numerical simulation is also demonstrated. The achieved results illuminate that, the future algorithm is easy to implement, highly methodical as well as effective and very accurate to analyse the behaviour of coupled nonlinear differential equations of fractional order arisen in the connected areas of science and engineering.

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Numerous pioneering directions are prescribed for the diverse definitions of fractional calculus by many senior researchers, and which prearranged the foundation \[1–6\]. Calculus with fractional order is associated to practical ventures and it extensively employed to nanotechnology, optics, human diseases, chaos theory, and other areas \[7–14\]. Particularly, FC becomes an essential tool in order to study and predict the hidden future of the interesting phenomena due to its ability to provide interesting properties like kernel, non-singularity and others, and many authors consider fractional calculus and presented some simulating results \[15–23\]. The numerical as well as analytical solution for these equations illustrating these models have an impairment role in portraying nature of nonlinear problems ascend in connected areas of science \[24–39\].

The systems of coupled KdV equations play a vibrating role in illustrating the interaction of distinct long waves having distinct dispersion relations. Which is effectively established by Hirota and Satsum in \[40,41\] and presented as follows

\[
u_t = \frac{1}{2} \left( u_{xxx} + 6 uu_x \right) = 2bv\nu_x, \\
u_t + 3 \nu_x + 3uv_x = 0.
\]  

(1)

Here \(2b\nu_x\) describes the forcing term on KdV system. Moreover, Drinfeld and Sokolov proved that this equation is a particular case of the four-reduced Kadomtsev-Petviashvili (KP) hierarchy \[42,43\]. Further, Wilson in \[44\] showed that, Eq. (1) is involved affine Lie algebras and it can be achieved as a general construction of Drinfel’d and Sokolov, and also illustrated how this equation associated to the affine Lie algebra \(\mathcal{C}_1^1\). Later, Wilson defined and exemplified the interesting equation, called Drinfeld–Sokolov–Wilson (DSW) equation and which as follows

\[
u_t + \eta \nu \nu_x = 0, \\
u_t + \nu x x x + \tau uv_x + \gamma u_x v = 0,
\]  

(2)

where \(u(x,t)\) and \(v(x,t)\) are the function of time \(t\) as well as space \(x\) and they designates the amplitude of the wave modes, \(\eta, \mu, \tau\) and \(\gamma\) are parameters and which are non-zero. Particularly, for \(\eta = \gamma = 1\) and \(\mu = \tau = 2\) the above system playas vital role in fluid mechanics and stimulating model of dispersive water waves.

In the present scenario, many important and nonlinear models are methodically and effectively analysed with the help of fractional calculus. There have been diverse definitions are suggested by many senior research scholars, for instance, Riemann, Liouville, Caputo and Fabrizio. However, these definitions have their own limitations. The Riemann–Liouville derivative is unable to explain the importance of the initial conditions; the Caputo derivative has overcome this shortcoming but is impotent to explain the singular kernel of the phenomena. Later, in 2015 Caputo and Fabrizio defeated the above obliges \[45\], and many researchers are considered this derivative in order to analyse and find the solution for diverse classes of nonlinear complex problems. But some issues were pointed out in CF derivative, like non-singular kernel and non-local, these properties are very essential in describing the physical behaviour and nature of the nonlinear problems. In 2016, Atangana and Baleanu introduced and matured the novel fractional derivative, namely AB derivative. This novel derivative defined with the aid of Mittag–Leffler functions \[46\]. This fractional derivative buried all the above-cited issues and helps us to understand the natural phenomena in the systematic and effective way.

In the present framework, we consider the fractional DSW (FDWS) equation of the form

\[
\frac{ABC}{a} D^a u(x,t) + 3v v_x = 0, \\
\frac{ABC}{a} D^a v(x,t) + 2v x x x + 2uv_x + u_x v = 0,
\]  

(3)

where \(x\) is fractional order of the system and defined with AB fractional operator. The fractional order is introduced in order to incorporate the memory effects and hereditary consequence in the system and these properties aids us to capture essential physical properties of the complex problems.

Recently, many mathematicians and physicists developed very effective and more accurate methods in order to find and analyse the solution for complex and nonlinear problems arisen in science and technology. In connection with this, the homotopy analysis method (HAM) proposed by Chinese Mathematician Liao Shijun \[47,48\]. HAM has been profitably and effectively applied to study the behaviour of nonlinear problems without perturbation or linearization. But for computational work, HAM requires huge time and computer memory. To overcome this, there is an essence of the amalgamation of considered method with well-known transform techniques.

In the present investigation, we put an effort to find and analysed the nature of the q-HATM solution for the FCDG equation by applying q-HATM. The future algorithm is the combination of q-HAM with LT \[49\]. Since q-HATM is an improved scheme of HAM; it does not require discretization, perturbation or linearization. Recently, due to its reliability and efficacy, the considered method is exceptionally applied by many researchers to understand physical behaviour diverse classes of complex problems \[50–57\]. The proposed method offers us more freedom to consider diverse class of initial guess and the equation type complex as well as nonlinear problems; because of this, the complex NDEs can be directly solved. The novelty of the future method is it aids a modest algorithm to evaluate the solution and it natured by the homotopy and axillary parameters, which provides the rapid convergence in the obtained solution for nonlinear portion of the given problem. Meanwhile, it has prodigious generality because it plausibly contains the results obtained by many algorithms like q-HAM, HPM, ADM and some other traditional techniques. The considered method can preserve great accuracy while decreasing the computational time and work in comparison with other methods.

The considered nonlinear coupled system recently magnetized the attention of researchers from different areas of science. Since DSW equation plays a significant role in portraying several complex phenomena, many authors find and analysed the solution using analytical as well as numerical schemes, for instance authors in \[58\] presented the conservation laws and also find the exact solutions for projected coupled system, the combination of Exp-Function and F-Expansion techniques are illustrated in \[59\] to find the solution, authors in \[60\] considered Sine-Gordon expansion and perturbation-iteration technique in order to find solution for system (2), Bäcklund transformation of Riccati equation is considered by the authors in \[61\] to find the exact solution for DSW system, authors in \[62\] presented conservation laws, Lie symmetry analysis and find the analytical solutions for the cited system. Further, many researchers investigated the DSW equation with the help of many algorithms \[63–67\]. In this
paper, we made an attempt to find the solution for FCSB equation using q-HATM.

2. Preliminaries

Recently, many authors considered these derivatives to analyse a diverse class of models in comparison with classical order as well as other fractional derivatives, and they prove that AB derivative is more effective while analysing the nature and physical behaviour of the models [68–71]. Here, we define the basic notion of Atangana-Baleanu derivatives and integrals [46].

**Definition 1.** The fractional Atangana-Baleanu-Caputo derivative for a function \( f \in H^l(a, b)(b > a, \alpha \in [0, 1]) \) is presented as follows

\[
\frac{ABC}{\alpha} D^\alpha_t (f(t)) = \frac{\mathcal{A}[f]}{1 - \frac{\alpha}{2}} \int_a^t f^{(\alpha)}(\theta) E_{\alpha} \left[ \frac{(t - \theta)^{\alpha}}{\alpha - 1} \right] d\theta.
\]  

**Definition 2.** The AB derivative of fractional order for a function \( f \in H^l(a, b), b > a, \alpha \in [0, 1] \) in Riemann-Liouville sense presented as follows

\[
\frac{ABC}{\alpha} D^\alpha_t (f(t)) = \frac{\mathcal{A}[f]}{1 - \frac{\alpha}{2}} \frac{d}{dt} \int_a^t f^{(\alpha)}(t - \theta)^{\alpha - 1} d\theta.
\]  

**Definition 3.** The fractional AB integral related to the non-local kernel is defined by

\[
\frac{ABC}{\alpha} I^\alpha_t (f(t)) = \frac{1}{\mathcal{A}[f]} f(t) + \frac{\alpha}{2} \frac{d}{dt} \int_a^t f^{(\alpha)}(t - \theta)^{\alpha - 1} d\theta.
\]  

**Definition 4.** The Laplace transform (LT) of AB derivative is defined by

\[
L\left[\frac{ABC}{\alpha} D^\alpha_t (f(t))\right] = \frac{\mathcal{A}[f]}{1 - \frac{\alpha}{2}} \frac{s^\alpha L[f(t)] - s^{\alpha - 1} f(0)}{s^\alpha + (\alpha / (1 - \alpha))}.
\]  

**Theorem 1.** The following Lipshitz conditions respectively hold true for both Riemann-Liouville and AB derivatives defined in Eqs. (4) and (5) [46],

\[
\|\frac{ABC}{\alpha} D^\alpha_t f_1(t) - \frac{ABC}{\alpha} D^\alpha_t f_2(t)\| < K_1 \|f_1(t) - f_2(t)\|,
\]  

and

\[
\|\frac{ABC}{\alpha} D^\alpha_t f_1(t) - \frac{ABC}{\alpha} D^\alpha_t f_2(t)\| < K_2 \|f_1(t) - f_2(t)\|.
\]

**Theorem 2.** The time-fractional differential equation \( a^\alpha BCD^\alpha_x f_1(t) = s(t) \) has a unique solution and which is defined as [46]

\[
f(t) = 1 - \frac{\alpha}{\mathcal{A}[s]} s(t) + \frac{\mu}{\mathcal{A}[s]} \int_0^t s(\tau)(t - \tau)^{\alpha - 1} d\tau.
\]  

3. Fundamental idea of the proposed scheme

Here, we consider the arbitrary order differential equation in order to demonstrate the basic solution procedure [72–75]}

\[
\frac{ABC}{\alpha} D^\alpha_x v(x, t) + \mathcal{A} v(x, t) + \mathcal{N} v(x, t) = f(x, t), \quad n - 1 < \alpha \leq n
\]

with the initial condition

\[
v(x, 0) = \phi(x)
\]

where \( \frac{ABC}{\alpha} D^\alpha_x v(x, t) \) symbolise the AB derivative of \( v(x, t) \) signifies the source term, \( \mathcal{A} \) and \( \mathcal{N} \) respectively denotes the linear and nonlinear differential operator. On using the LT on Eq. (11), we have after simplification

\[
\mathcal{L}[v(x, t)] = \frac{\phi(x)}{s} + \frac{1}{\mathcal{A}[\mathcal{L}]} \left( 1 - \alpha + \frac{s}{\alpha} \right) \left( \mathcal{L}[\mathcal{A} v(x, t)] + \mathcal{L}[\mathcal{N} v(x, t)] - \mathcal{L}[f(x, t)] \right) = 0
\]

The non-linear operator is presented as

\[
\mathcal{N} [\phi(x, t; q)] = \mathcal{L}[\phi(x, t; q)] - \frac{\phi(x)}{s}
\]

\[
+ \frac{1}{\mathcal{A}[\mathcal{L}]} \left( 1 - \alpha + \frac{s}{\alpha} \right) \left( \mathcal{L}[\mathcal{A} \phi(x, t; q)] + \mathcal{L}[\mathcal{N} \phi(x, t; q)] - \mathcal{L}[f(x, t)] \right)
\]

where, \( \phi(x, t; q) \) is the real valued function with respect to \( x, \text{ and } q \in [0, \frac{1}{2}] \). Now, we define a homotopy as follows

\[
(1 - q)\mathcal{L}[\phi(x, t; q) - v_0(0, t)] = hq\mathcal{L}[\phi(x, t; q)]
\]

where \( h \) signifies the embedding parameter and \( h \neq 0 \) is an auxiliary parameter. For \( q = 0 \) and \( q = \frac{1}{n} \) the results given below are hold true

\[
\phi(x, t; 0) = v_0(x, t), \quad \phi\left(x, t; \frac{1}{n}\right) = v(x, t)
\]

Now, by intensifying \( q \) from \( 0 \) to \( \frac{1}{n} \), then \( \phi(x, t; q) \) varies from \( v_0(x, t) \) to \( v(x, t) \). By using the Taylor theorem near \( q \), we define \( \phi(x, t; q) \) in series form and then we get

\[
\phi(x, t; q) = v_0(x, t) + \sum_{n=1}^{\infty} v_n(x, t) q^n
\]

where

\[
v_n(x, t) = \frac{1}{n!} \frac{\partial^n \phi(x, t; q)}{\partial q^n} \bigg|_{q=0}
\]

The series (14) converges at \( q = \frac{1}{n} \) for the proper chaise of \( v_0(x, t), n \) and \( h \). Then

\[
v(x) = v_0(x, t) + \sum_{n=1}^{m} v_m(x, t) \left( \frac{1}{n} \right)^m
\]

On \( m \)-times differentiating Eq. (15) with \( q \) and finally dividing by \( ml \) and then substituting \( q = 0 \), we get

\[
\mathcal{L}[v_m(x, t) - k_m v_{m-1}(x, t)] = h\mathcal{R}_m \left( \mathcal{V}_{m-1} \right)
\]

where the vectors are defined as

\[
\mathcal{V}_m = \{v_0(x, t), v_1(x, t), \ldots, v_m(x, t)\}.
\]

On employing the inverse LT on Eq. (20), we have

\[
v_m(x, t) = k_m v_{m-1}(x, t) + h\mathcal{R}_{m-1}^{-1} \left[ \mathcal{R}_m \left( \mathcal{V}_{m-1} \right) \right]
\]

where
For DSW equation, the analytical solution is presented as follows

\[
\mathcal{H}_m = \frac{1}{m!} \left[ \frac{\partial^n \varphi(x; t; q)}{\partial q^n} \right]_{q=0}
\]

where

\[
\varphi_0 = q \varphi_1 + q^2 \varphi_2 + \cdots
\]

By the aid of Eqs. (22) and (23), one can get

\[
v_m(x, t) = (k_m + \mathcal{H}) v_m-1(x, t)
\]

Then, the terms of \(v_m(x, t)\) we can obtain using the Eq. (26). The \(q\)-HATM series solution is presented as

\[
v(x, t) = \sum_{n=0} v_n(x, t).
\]

4. Solution for FDSW equation

In order to present the solution procedure and efficiency of the future scheme, in this segment we consider DSW equation of fractional order with two distinct cases. Further, by the help of obtained results we made an attempt to capture the behaviour of \(q\)-HATM solution for different fractional order. By the help of Eq. (3), we have

\[
a^\beta \partial^\alpha_x u(x, t) + 3v_x = 0, \quad a^\beta \partial^\alpha_x v(x, t) + 2v_{xxx} + 2v_x + u_v = 0,
\]

with initial conditions

\[
\begin{align*}
u(x, 0) &= 3\text{sech}^2(x), \\
v(x, 0) &= 2\text{sech}(x).
\end{align*}
\]

For DSW equation, the analytical solution is presented as follows

\[
u(x, t) = \frac{3c}{2} \text{sech}^2 \left( \frac{c}{2} (x - ct) \right).
\]

\[
v(x, t) = c \text{sech} \left( \frac{c}{2} (x - ct) \right).
\]

Taking \(LT\) on Eq. (28) and then using the Eq. (29), we get

\[
L[u(x, t) - \frac{1}{4} (3\text{sech}^2(x)) + \int_{\mathbb{R}} (1 - x + \frac{z}{s}) L \left\{ \nu_m^{(m)} \right\} = 0
\]

\[
L[v(x, t)] - \frac{1}{2} (2\text{sech}(x)) + \int_{\mathbb{R}} (1 - x + \frac{z}{s}) L \left\{ \nu_m^{(m)} \right\} = 0.
\]

The non-linear operator \(N\) is presented with the help of future algorithm as below

\[
N[\varphi_1(x, t; q), \varphi_2(x, t; q)] = L[\varphi_1(x, t; q)] - \frac{1}{s} (3\text{sech}^2(x))
\]

\[
N^2[\varphi_1(x, t; q), \varphi_2(x, t; q)] = L[\varphi_2(x, t; q)] - \frac{1}{s} (2\text{sech}(x))
\]

The deformation equation of \(m\)-th order by the help of \(q\)-HATM at \(\mathcal{H}(x, t) = 1\), is given as follows

\[
L[u_m(x, t) - k_m u_{m-1}(x, t)] + \mathcal{H} L[u_m(x, t - k_m u_m(x, t)]
\]

where

\[
u_{m-1}(x, t) = L[u_m(x, t) - k_m u_{m-1}(x, t)]
\]

On applying inverse \(LT\) on Eq. (32), it reduces to

\[
u_m(x, t) = k_m u_{m-1}(x, t) + \text{h} L^{-1} \{ \mathcal{H} L[u_m(x, t) - k_m u_{m-1}(x, t)] \}
\]

\[
u_m(x, t) = k_m u_{m-1}(x, t) + \text{h} L^{-1} \{ \mathcal{H} L[u_m(x, t) - k_m u_{m-1}(x, t)] \}
\]

On simplifying the above equation systematically by using \(u_m(x, t) = 3\text{sech}^2(x)\) and \(v_0(x, t) = 2\text{sech}(x)\) we obtained the terms of the series solution

\[
u(x, t) = u_0(x, t) + \sum_{n=1}^{\infty} u_n(x, t) \left( \frac{c}{2} \right)^n
\]

\[
v(x, t) = v_0(x, t) + \sum_{n=1}^{\infty} v_n(x, t) \left( \frac{c}{2} \right)^n.
\]

5. Existence of solutions for the future model

Here, we considered the fixed-point theorem in order to demonstrate the existence of the solution for the proposed model. Since, the considered model cited in the system (28) is non-local as well as complex; there are no particular algo-
Theorem 3. The kernel \( \mathcal{G}_1 \) satisfies the Lipschitz condition and contraction if the condition \( 0 \leq (\delta^3 + (2 + \delta)\lambda_1) < 1 \) holds.

Table 1 Comparison of the obtained solution with HASTM \([42]\) in the form of absolute error for FDSWE equation at \( c = 2, n = 1, h = -1 \) and \( x = 1 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \delta u^{(1)}<em>{\text{Exact}} - \delta u^{(1)}</em>{\text{HASTM}} )</th>
<th>( \delta u^{(2)}<em>{\text{Exact}} - \delta u^{(2)}</em>{\text{HASTM}} )</th>
<th>( \delta u^{(3)}<em>{\text{Exact}} - \delta u^{(3)}</em>{\text{HASTM}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>1.813 \times 10^{-6}</td>
<td>1.81285 \times 10^{-6}</td>
<td>2.37318 \times 10^{-10}</td>
</tr>
<tr>
<td>0.04</td>
<td>2.9551 \times 10^{-5}</td>
<td>2.95504 \times 10^{-5}</td>
<td>1.70277 \times 10^{-8}</td>
</tr>
<tr>
<td>0.06</td>
<td>1.25294 \times 10^{-4}</td>
<td>1.25229 \times 10^{-4}</td>
<td>2.15729 \times 10^{-7}</td>
</tr>
<tr>
<td>0.08</td>
<td>4.8959 \times 10^{-3}</td>
<td>4.89596 \times 10^{-4}</td>
<td>1.33893 \times 10^{-6}</td>
</tr>
<tr>
<td>0.1</td>
<td>1.214764 \times 10^{-3}</td>
<td>1.21476 \times 10^{-3}</td>
<td>5.60791 \times 10^{-6}</td>
</tr>
</tbody>
</table>

| \( v^{(1)}_{\text{Exact}} - v^{(1)}_{\text{HASTM}} \) | \( v^{(2)}_{\text{Exact}} - v^{(2)}_{\text{HASTM}} \) | \( v^{(3)}_{\text{Exact}} - v^{(3)}_{\text{HASTM}} \) |
|-------|---------------------------------|---------------------------------|---------------------------------|
| 0.02  | 4.40 \times 10^{-7}             | 4.40168 \times 10^{-7}         | 1.47464 \times 10^{-10}        |
| 0.04  | 7.076 \times 10^{-6}            | 7.07619 \times 10^{-6}         | 9.72998 \times 10^{-9}         |
| 0.06  | 3.5964 \times 10^{-5}           | 3.59652 \times 10^{-5}         | 1.14202 \times 10^{-7}         |
| 0.08  | 1.14022 \times 10^{-4}          | 1.14022 \times 10^{-4}         | 6.60794 \times 10^{-7}         |
| 0.1   | 2.78991 \times 10^{-4}          | 2.78990 \times 10^{-4}         | 2.59428 \times 10^{-6}         |

Table 2 Numerical simulation presented for \( u(x, t) \) of FDSWE equation at \( c = 2, n = 1, h = -1 \) and \( x = 1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( t )</th>
<th>( \delta u^{(2)}<em>{\text{Exact}} - \delta u^{(2)}</em>{\text{HASTM}} )</th>
<th>( \delta u^{(3)}<em>{\text{Exact}} - \delta u^{(3)}</em>{\text{HASTM}} )</th>
<th>( \delta u^{(4)}<em>{\text{Exact}} - \delta u^{(4)}</em>{\text{HASTM}} )</th>
<th>( \delta u^{(5)}<em>{\text{Exact}} - \delta u^{(5)}</em>{\text{HASTM}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0.025</td>
<td>1.23436 \times 10^{-5}</td>
<td>2.71943 \times 10^{-7}</td>
<td>4.07243 \times 10^{-9}</td>
<td>2.53013 \times 10^{-11}</td>
</tr>
<tr>
<td>0.05</td>
<td>0.075</td>
<td>1.00990 \times 10^{-4}</td>
<td>4.41701 \times 10^{-6}</td>
<td>1.31075 \times 10^{-7}</td>
<td>1.71942 \times 10^{-8}</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>8.45395 \times 10^{-5}</td>
<td>2.26982 \times 10^{-5}</td>
<td>1.00065 \times 10^{-6}</td>
<td>9.26078 \times 10^{-8}</td>
</tr>
<tr>
<td>5</td>
<td>0.025</td>
<td>9.30466 \times 10^{-8}</td>
<td>2.31273 \times 10^{-9}</td>
<td>4.60312 \times 10^{-11}</td>
<td>7.63089 \times 10^{-13}</td>
</tr>
<tr>
<td>0.05</td>
<td>0.075</td>
<td>7.63637 \times 10^{-7}</td>
<td>3.77653 \times 10^{-8}</td>
<td>1.49813 \times 10^{-9}</td>
<td>4.95475 \times 10^{-11}</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>6.44699 \times 10^{-6}</td>
<td>1.95175 \times 10^{-7}</td>
<td>1.15728 \times 10^{-8}</td>
<td>5.72667 \times 10^{-10}</td>
</tr>
<tr>
<td>7.5</td>
<td>0.025</td>
<td>6.27408 \times 10^{-10}</td>
<td>1.56060 \times 10^{-13}</td>
<td>3.11068 \times 10^{-13}</td>
<td>5.17192 \times 10^{-15}</td>
</tr>
<tr>
<td>0.05</td>
<td>0.075</td>
<td>5.14926 \times 10^{-9}</td>
<td>2.54844 \times 10^{-10}</td>
<td>1.01245 \times 10^{-11}</td>
<td>3.35843 \times 10^{-13}</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>4.34062 \times 10^{-8}</td>
<td>1.31711 \times 10^{-9}</td>
<td>7.82148 \times 10^{-10}</td>
<td>3.88202 \times 10^{-12}</td>
</tr>
<tr>
<td>10</td>
<td>0.025</td>
<td>4.22746 \times 10^{-12}</td>
<td>1.05154 \times 10^{-13}</td>
<td>2.09601 \times 10^{-15}</td>
<td>3.48495 \times 10^{-17}</td>
</tr>
<tr>
<td>0.05</td>
<td>0.075</td>
<td>3.46956 \times 10^{-11}</td>
<td>1.71714 \times 10^{-12}</td>
<td>6.82199 \times 10^{-14}</td>
<td>2.26299 \times 10^{-15}</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>2.92470 \times 10^{-10}</td>
<td>8.87469 \times 10^{-12}</td>
<td>5.27018 \times 10^{-13}</td>
<td>2.61579 \times 10^{-14}</td>
</tr>
</tbody>
</table>

Proof. In order to prove the required result, we consider the two functions \( S \) and \( S_1 \), then

\[
\| \mathcal{G}_2(x, t, v) - \mathcal{G}_2(x, t, v) \| = \| 2 \frac{\partial^3}{\partial x^3} [v(x, t) - v(x, t)] + 2u(x, t) \frac{\partial}{\partial x} [v(x, t) - v(x, t)] + \frac{\partial u(x, t)}{\partial x} [v(x, t) - v(x, t)] \|
\]

\[
\leq \| 2 \delta^3 + 2u(x, t) \|
\]

The recursive form of Eq. (37) defined as follows

\[
u_a(x; t, u_{a+1}) = (1 + \delta) \mathcal{G}_2(x, t, u_{n-1}) + \frac{\partial}{\partial t} \mathcal{G}_1(x, t, u_{n-1}) (t - \zeta)^{\delta-1} d\zeta.
\]

The associated initial conditions are
Theorem 4. The solution for the system (28) will exist and unique if we have specific to then

\[
\frac{(1 - z)}{\mathcal{B}(z)} \eta_1 + \frac{z}{\mathcal{B}(z) \Gamma(z)} \eta_2 < 1,
\]

for \( i = 1 \) and \( 2 \).

Proof. Let us consider the bounded functions \( u(x,t) \) and \( v(x,t) \) satisfying the Lipschitz condition. Then, by Eqs. (44) and (46), we have

\[
\| \phi_1(x,t) \| \leq \| u_0(x,0) \| \left( \frac{(1 - z)}{\mathcal{B}(z)} \eta_1 + \frac{z}{\mathcal{B}(z) \Gamma(z)} \eta_2 \right)
\]

\[
\| \phi_2(x,t) \| \leq \| v_0(x,0) \| \left( \frac{(1 - z)}{\mathcal{B}(z)} \eta_2 + \frac{z}{\mathcal{B}(z) \Gamma(z)} \eta_1 \right).
\]

Therefore, the continuity as well as existence for the obtained solutions is proved. Subsequently, in order to show the system (47) is a solution for the system (29), we consider

\[
u(x,0) - u(x,0) = u_n(x,t) - u(x,0) \quad \text{and} \quad v(x,0) - v(x,0) = v_n(x,t) - v(x,0).
\]

In order to obtain a result, we consider

\[
\| \mathcal{K}_n(x,t) \| = \| \left( \frac{(1 - z)}{\mathcal{B}(z)} \mathcal{G}(x,t,u) - \mathcal{G}(x,t,u_n-1) \right) \|
\]

\[
\leq \frac{(1 - z)}{\mathcal{B}(z)} \mathcal{G}(x,t,u) + \frac{z}{\mathcal{B}(z) \Gamma(z)} \eta_2 \| u_n-1 \|
\]

\[
\leq \left( \frac{1 - z}{\mathcal{B}(z)} \right) \| u - u_n-1 \| + \frac{z}{\mathcal{B}(z) \Gamma(z)} \eta_2 \| u - u_n-1 \|. \quad (49)
\]
A powerful approach for fractional Drinfeld–Sokolov–Wilson equation with Mittag-Leffler law

Fig. 1 Surfaces of (a) $u_{q\text{-HATM}}$, (b) $u_{Exact}$, (c) $|u_{Exact} - u_{q\text{-HATM}}|$, (d) $v_{q\text{-HATM}}$, (e) $v_{Exact}$, (f) $|v_{Exact} - v_{q\text{-HATM}}|$ at $c = 2$, $h = -1$, $n = 1$ and $z = 1$. 
Similarly, at \( t_0 \) we can obtain
\[
\| \mathbf{X}(x, t) \| \leq \left( \frac{1 - \alpha}{\mathcal{B}(x)} + \frac{\alpha t_0}{\mathcal{B}(x) \Gamma(x)} \right)^{s+1} \eta_1^{s+1} M. \tag{50}
\]

As \( \eta_1 \) approaches to \( \infty \), we can see that form Eq. (50), \( \| \mathbf{X}(x, t) \| \) tends to 0. Similarly, we can verify for \( \| \mathbf{X}(x, t) \| \). Next, it is necessary to demonstrate uniqueness for the solution of the considered model. Suppose, \( u'(x, t) \) and \( v'(x, t) \) be the set of other solutions, then we have
\[
u(x, t) - u'(x, t) = \left( \frac{1 - \alpha}{\mathcal{B}(x)} \right) (\mathcal{P}(x, t, u) - \mathcal{P}(x, t, u')) + \frac{\alpha}{\mathcal{B}(x) \Gamma(x)} \int_0^t (\mathcal{P}(x, \zeta, u) - \mathcal{P}(x, \zeta, u')) \, d\zeta.
\]

On applying norm, the Eq. (51) simplifies to
\[
\| u(x, t) - u'(x, t) \| \leq \left( \frac{1 - \alpha}{\mathcal{B}(x)} \right) \eta_1 \| u(x, t) - u'(x, t) \| + \frac{\alpha}{\mathcal{B}(x) \Gamma(x)} \eta_1 \| u(x, t) - u'(x, t) \|. \tag{52}
\]

On simplification
\[
\| u(x, t) - u'(x, t) \| \left( 1 - \frac{(1 - \alpha)}{\mathcal{B}(x)} \eta_1 - \frac{\alpha}{\mathcal{B}(x) \Gamma(x)} \eta_1 t \right) \leq 0. \tag{53}
\]

From the above condition, it is clear that \( u(x, t) - u'(x, t) \), if
\[
\left( 1 - \frac{(1 - \alpha)}{\mathcal{B}(x)} \eta_1 - \frac{\alpha}{\mathcal{B}(x) \Gamma(x)} \eta_1 t \right) \geq 0. \tag{54}
\]

Hence, Eq. (54) evidences our essential result. \( \Box \)

6. Numerical results and discussion

In the present investigation, we find the solution for DSW equation having arbitrary order using a novel scheme namely, \( q \)-HATM with the help of Mittag-Leffler law. Here, we demonstrate the numerical simulation for the considered coupled system. In Table 1, the comparison of the obtained solution with HASTM is presented and in Tables 2 and 3, the error analysis has been validated. From the tables we can see that, the proposed scheme is more accurate and HASTM is a special case of \( q \)-HATM (i.e., \( n = 1 \)) and we confirm that as the number of iteration \( s \) increases the \( q \)-HATM solutions get converges to the exact solution. Also, as time increases the accuracy of the obtained solution decreases this shows that the proposed model highly depends on time instant, and hence we can analyse with the help of time-fractional derivatives.

The surfaces of the obtained solution and the exact solution in comparison with absolute error have been captured in Fig. 1. The behaviour of \( q \)-HATM solution for the coupled system is cited in Fig. 2 understand the physical behaviour of the nonlinear coupled system. The behaviour of the obtained solution for different order is presented in Fig. 3 in terms of 2D plots. In order to analyse the variations of the obtained solution with respect to homotopy parameter (\( \alpha \)), the \( \alpha \)-curves are drown for diverse \( \alpha \) with distinct \( c = 2, h = -1, n = 1 \) and \( x = 1 \).
The consequences of the obtained solution are presented in terms of plots in order to understand the physical behaviour of the considered nonlinear coupled system. From the plots we confirm the exactness of the proposed method. Further, in order to show the computational level of the future method, we presented numerical simulation in the tables. The small deviation in the physical behaviour of the complex models stimulates the enormous new results to analyse and understand nature in a better and systematic manner. Moreover, from all the plots we can see that, the proposed method is more accurate and very effective to analyse the considered coupled fractional order equations.

7. Conclusion

In this paper, the $q$-HATM is applied profitably to find the solution for an arbitrary order DSW equation. Since AB derivatives and integrals having fractional order are defined with the help of generalized Mittag-Leffler function as the non-local kernel and non-singular, the present investigation illuminates the effeteness of the considered derivative. The existence and uniqueness of the obtained solution are demonstrated with the fixed point hypothesis. The results obtained by the future scheme are more stimulating as compared to results available in the literature. Further, the proposed algorithm finds the solution of the coupled nonlinear problem without considering any discretization, perturbation or transformation. The present investigation illuminates, the considered complex nonlinear phenomena noticeably depend on the time history and the time instant and which can be proficiently analysed by applying the concept of calculus with fractional order. The present investigation helps the researchers to study the behaviour nonlinear problems gives very interesting and useful consequences. Lastly, we can conclude the projected method is extremely methodical, more effective and very accurate, and which can be applied to analyse the diverse classes of coupled nonlinear problems exist in science and technology.

Declaration of Competing Interest

The authors declared that there is no conflict of interest.

References

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